

LIMIT AND END FUNCTORS OF DYNAMICAL SYSTEMS VIA EXTERIOR SPACES

J.M. GARCÍA CALCINES, L. HERNÁNDEZ PARICIO AND M. TERESA RIVAS
RODRÍGUEZ

ABSTRACT. In this paper we analyze some applications of the category of exterior spaces to the study of dynamical systems (flows). We study the notion of an absorbing open subset of a dynamical system; i.e., an open subset that contains the “future part” of all the trajectories. The family of all absorbing open subsets is a quasi-filter which gives the structure of an exterior space to the flow. The limit space and end space of an exterior space is used to construct the limit spaces and end spaces of a dynamical system. On the one hand, for a dynamical system two limits spaces $L^r(X)$ and $\bar{L}^r(X)$ are constructed and their relations with the subflows of periodic, Poisson stable points and Ω^r -limits of X are analyzed. On the other hand, different end spaces are also associated to a dynamical system having the property that any positive semi-trajectory has an end point in these end spaces. This type of construction permits us to consider the subflow containing all trajectories finishing at an end point a . When a runs over the set of all end points, we have an induced decomposition of a dynamical system as a disjoint union of stable (at infinity) subflows.

1. INTRODUCTION

Many natural phenomena can be modeled by means of a family of differential equations and each one can be put (maybe after some manipulations) in the form $\dot{\phi} = f(\phi)$, where ϕ are local coordinates in an open neighborhood of the point $p \in M$, where M is an m -dimensional manifold, $\dot{\phi}$ are the coordinates of tangent vectors in the open neighborhood of the point $p \in M$ and f is a real valued function whose domain is an open of \mathbb{R}^m .

Under the assumption of f being locally lipschitzian, an initial condition $\phi^p(0) = p$ uniquely determines a maximal solution $\phi^p(t)$. However, the domain of $\phi^p(t)$ does not need to be the whole real line \mathbb{R} , but only an open interval, $(a^p, b^p) \subset \mathbb{R}$, $a^p < 0 < b^p$, which depends on the initial condition. All the solutions give a local flow $\phi: W \rightarrow M$, $\phi(t, p) = \phi^p(t)$, where W is an open subset of $\mathbb{R} \times M$ containing $\{0\} \times M$ and if we denote $\phi_s(p) = \phi(s, p)$, $(s, p) \in W$, ϕ satisfies $\phi_0 = \text{id}_M$, $\phi_t \phi_s = \phi_{t+s}$, wherever it makes sense. The space M is called the phase space and ϕ is also called the phase map. The trajectory of a point $p \in M$ is the subset $\gamma(p) = \{\phi(t, p) | t \in (a^p, b^p)\}$. It is easy to check that M is a disjoint union of trajectories. We note that when a trajectory has more than one point, a natural

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orientation is induced by the canonical orientation of the real numbers \mathbb{R} . Then, we can consider M as a disjoint union of critical trajectories and oriented trajectories to obtain a phase portrait of the dynamical system ϕ . It is well known that (under some assumptions, see [3]) if φ is a local flow on M , then there exists a global ($W = \mathbb{R} \times M$) flow ϕ in M such that the oriented trajectories of φ and ϕ coincide. Consequently their phase portraits are the same. As a consequence of this type of result we have considered convenient to reduce our study to the case of global flows. This allows us a functorial approach to the study of dynamical systems and their properties.

On the other side, for every topological space X , a continuous map $\varphi: \mathbb{R} \times X \rightarrow X$ induces a group homomorphism $\bar{\varphi}: \mathbb{R} \rightarrow \text{Aut}(X)$, where $\text{Aut}(X)$ is the group of homeomorphisms of X provided with the compact-open topology. This fact permits to study a flow as a particular case of a transformation group.

The pioneering work of H. Poincaré [22, 23] in the late XIX century studied the topological properties of solutions of autonomous ordinary differential equations. We can also mention the work of A. M. Lyapunov [15] who developed his theory of stability of a motion (solution) of a system of n first order ordinary differential equations. While much of Poincaré's work analyzed the global properties of the system, Lyapunov's work looks at the local stability of a dynamical system. The theory of dynamical systems reached a great development with the work of G.D. Birkhoff [2], who may be considered as the founder of this theory. He established two main lines in the study of dynamical systems: the topological theory and the ergodic theory.

In this paper, we describe some basic ideas that permit a new treatment of the study of dynamical system. Previously, the authors have developed some results on proper homotopy theory and exterior homotopy theory to classify non compact spaces and to study the shape of a compact metric space. In this work, the main objective proposed by the authors is to describe functorially a way of using exterior spaces to study dynamical systems. In our approach, the main key to establish a connection between exterior spaces and dynamical systems is the notion of absorbing open region (or \mathbf{r} -exterior subset). Given a flow on space X , an open set E is said to be \mathbf{r} -exterior if for every $x \in X$ there is $r_0 \in \mathbb{R}$ such that $r \cdot x \in E$ for $r \geq r_0$. The space X together with the family of \mathbf{r} -exterior open subsets is an exterior space which is denoted by $X^{\mathbf{r}}$.

On the one hand, using exterior spaces, we construct two limit subflows $L^{\mathbf{r}}(X)$ and $\bar{L}^{\mathbf{r}}(X)$ associated with a flow X . One of the important results of our study (see Corollary 6.14) is the following:

If X is a locally compact T_3 space, then $L^{\mathbf{r}}(X) = P(X)$ and $\bar{L}^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$; furthermore we have that

$$L^{\mathbf{r}}(X) = P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X) \subset \overline{\Omega^{\mathbf{r}}(X)} = \bar{L}^{\mathbf{r}}(X),$$

where $P(X)$ is the subflow of periodic points, $P^{\mathbf{r}}(X)$ is the subflow of right (positive) Poisson stable points and $\Omega^{\mathbf{r}}(X) = \bigcup_{x \in X} \omega^{\mathbf{r}}(x)$ ($\omega^{\mathbf{r}}(x)$ is the right limit set of the point $x \in X$). Under the condition of being \mathbf{r} -regular at infinity (see Proposition 3.14), we have that $L^{\mathbf{r}}(X) = \bar{L}^{\mathbf{r}}(X)$ and in some cases we can also ensure that $\bar{L}^{\mathbf{r}}(X)$ is compact. The constructions $L^{\mathbf{r}}, \bar{L}^{\mathbf{r}}$ induce classifications (for flows) of the following type: Two flows X, Y are said to be \bar{L} -equivalent if there is a flow morphism $f: X \rightarrow Y$ such that $\bar{L}(f)$ is a homotopy equivalence. This will determine

equivalence classes of flows having, for instance, the same number of critical points or the ‘same type’ of periodic trajectories.

On the other hand, using again exterior spaces, we construct ‘slightly different’ end spaces $\tilde{\pi}_0^{\mathbf{r}}(X)$, $\tilde{c}^{\mathbf{r}}(X)$, $\tilde{\pi}_0^{\mathbf{r}}(X)$, $\tilde{c}^{\mathbf{r}}(X)$, which agree when X is locally path-connected and \mathbf{r} -regular at infinity. In this case, the end spaces have a pro-discrete topology (with additional conditions, a pro-finite topology). The importance of this end space is that each right semi-trajectory of the flow has an end point in this space. This fact permits to give a set map $\bar{\omega}_{\mathbf{r}}: X \rightarrow \tilde{c}^{\mathbf{r}}(X)$ and the corresponding $\bar{\omega}_{\mathbf{r}}$ -decomposition of the flow

$$X = \bigsqcup_{a \in \tilde{c}^{\mathbf{r}}(X)} \bar{X}_{(\mathbf{r}, a)}.$$

In general, there are end points which are not reached by right semi-trajectories and there are end points of semi-trajectories that are not reached by right semi-trajectories contained in the limit subflow. In this paper, we give sufficient conditions to ensure that an end point can be reached by right semi-trajectories of $\bar{L}(X)$ (see Proposition 4.8). These end spaces will be used (not in this work) to construct completions (compactification under some topological and dynamical conditions) of a flow. These completions are related to Freudenthal compactifications [6, 8, 14] and will permit to apply some nice properties of compact flows to a more general class of topological flows.

It is important to observe that applying the results of this paper to the reversed flow we will obtain all the corresponding dual concepts, constructions and properties.

2. PRELIMINARIES ON EXTERIOR SPACES AND DYNAMICAL SYSTEMS

2.1. The categories of proper and exterior spaces. A continuous map $f: X \rightarrow Y$ is said to be proper if for every closed compact subset K of Y , $f^{-1}(K)$ is a compact subset of X . The category of topological spaces and the subcategory of spaces and proper maps will be denoted by **Top** and **P**, respectively. This last category and its corresponding proper homotopy category are very useful for the study of non compact spaces. Nevertheless, one has the problem that **P** does not have enough limits and colimits and then we can not develop the usual homotopy constructions such as loops, homotopy limits and colimits, et cetera. An answer to this problem is given by the notion of exterior space. The new category of exterior spaces and maps is complete and cocomplete and contains as a full subcategory the category of spaces and proper maps, see [9, 11]. For more properties and applications of exterior homotopy categories we refer the reader to [10, 7, 4, 12, 13] and for a survey of proper homotopy to [21].

Definition 2.1. Let (X, \mathbf{t}) be a topological space, where X is the subjacent set and \mathbf{t} its topology. An *externology* on (X, \mathbf{t}) is a non empty collection ε (also denoted by $\varepsilon(X)$) of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \mathbf{t}$ and $E \subset U$ then $U \in \varepsilon$. The members of ε are called *exterior open subsets*. An *exterior space* $(X, \varepsilon, \mathbf{t})$ consists of a space (X, \mathbf{t}) together with an *externology* ε . A map $f: (X, \varepsilon, \mathbf{t}) \rightarrow (X', \varepsilon', \mathbf{t}')$ is said to be an *exterior map* if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.

The category of exterior spaces and maps will be denoted by **E**. Given an space (X, \mathbf{t}_X) , we can always consider the trivial exterior space taking $\varepsilon = \{X\}$ or the

total exterior space if one takes $\varepsilon = \mathbf{t}_X$. An important example of externology on a given topological space X is the one constituted by the complements of all closed-compact subsets of X , that will be called the cocompact externology and usually written as $\varepsilon^c(X)$. The category of spaces and proper maps can be considered as a full subcategory of the category of exterior spaces via the full embedding $(\cdot)^c : \mathbf{P} \hookrightarrow \mathbf{E}$. The functor $(\cdot)^c$ carries a space X to the exterior space X^c which is provided with the topology of X and the externology $\varepsilon^c(X)$. A map $f : X \rightarrow Y$ is carried to the exterior map $f^c : X^c \rightarrow Y^c$ given by $f^c = f$. It is easy to check that a continuous map $f : X \rightarrow Y$ is proper if and only if $f = f^c : X^c \rightarrow Y^c$ is exterior.

An important role in this paper will be played by the following construction $(\cdot) \bar{\times} (\cdot)$: Let $(X, \varepsilon(X), \mathbf{t}_X)$ be an exterior space, (Y, \mathbf{t}_Y) a topological space and for $y \in Y$ we denote by $(\mathbf{t}_Y)_y$ the family of open neighborhoods of Y at y . We consider on $X \times Y$ the product topology $\mathbf{t}_{X \times Y}$ and the externology $\varepsilon(X \bar{\times} Y)$ given by those $E \in \mathbf{t}_{X \times Y}$ such that for each $y \in Y$ there exists $U_y \in (\mathbf{t}_Y)_y$ and $T^y \in \varepsilon(X)$ such that $T^y \times U_y \subset E$. This exterior space will be denoted by $X \bar{\times} Y$ in order to avoid a possible confusion with the product externology. This construction gives a functor

$$(\cdot) \bar{\times} (\cdot) : \mathbf{E} \times \mathbf{Top} \rightarrow \mathbf{E}.$$

When Y is a compact space, we have that E is an exterior open subset if and only if it is an open subset and there exists $G \in \varepsilon(X)$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon(X) = \varepsilon^c(X)$, then $\varepsilon(X \bar{\times} Y)$ coincides with $\varepsilon^c(X \times Y)$ the externology of the complements of closed-compact subsets of $X \times Y$. We also note that if Y is a discrete space, then E is an exterior open subset if and only if it is open and for each $y \in Y$ there is $T^y \in \varepsilon(X)$ such that $T^y \times \{y\} \subset E$.

This bar construction provides a natural way to define *exterior homotopy* in \mathbf{E} . Indeed, if I denotes the closed unit interval, given exterior maps $f, g : X \rightarrow Y$, it is said that f is *exterior homotopic* to g if there exists an exterior map $H : X \bar{\times} I \rightarrow Y$ (called exterior homotopy) such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, for all $x \in X$. The corresponding homotopy category of exterior spaces will be denoted by $\pi\mathbf{E}$. Similarly, the usual homotopy category of topological spaces will be denoted by $\pi\mathbf{Top}$.

2.2. Dynamical Systems and Ω -Limits. Next we recall some elementary concepts about dynamical systems.

Definition 2.2. A dynamical system (or a flow) on a topological space X is a continuous map $\varphi : \mathbb{R} \times X \rightarrow X$, $\varphi(t, x) = t \cdot x$, such that

- (i) $\varphi(0, p) = p$, $\forall p \in X$;
- (ii) $\varphi(t, \varphi(s, p)) = \varphi(t + s, p)$, $\forall p \in X$, $\forall t, s \in \mathbb{R}$.

A flow on X will be denoted by (X, φ) and when no confusion is possible, we use X for short.

For a subset $A \subset X$, we denote $\text{inv}(A) = \{p \in A \mid \mathbb{R} \cdot p \subset A\}$.

Definition 2.3. A subset S of a flow X is said to be invariant if $\text{inv}(S) = S$.

Given a flow $\varphi : \mathbb{R} \times X \rightarrow X$ one has a subgroup $\{\varphi_t : X \rightarrow X \mid t \in \mathbb{R}\}$ of homeomorphisms, $\varphi_t(x) = \varphi(t, x)$, and a family of motions $\{\varphi^p : \mathbb{R} \rightarrow X \mid p \in X\}$, $\varphi^p(t) = \varphi(t, p)$.

Definition 2.4. Given two flows $\varphi: \mathbb{R} \times X \rightarrow X$, $\psi: \mathbb{R} \times Y \rightarrow Y$, a flow morphism $f: (X, \varphi) \rightarrow (Y, \psi)$ is a continuous map $f: X \rightarrow Y$ such that $f(r \cdot p) = r \cdot f(p)$ for every $r \in \mathbb{R}$ and for every $p \in X$.

We note that if $S \subset X$ is invariant, S has a flow structure and the inclusion is a flow morphism.

We denote by \mathbf{F} the category of flows and flows morphisms.

We recall some basic fundamental examples: (1) $X = \mathbb{R}$ with the action $\varphi: \mathbb{R} \times X \rightarrow X$, $\varphi(r, s) = r + s$. (2) $X = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with $\varphi: \mathbb{R} \times X \rightarrow X$, $\varphi(r, z) = e^{2\pi i r} z$. (3) $X = \{0\}$ with the trivial action $\varphi: \mathbb{R} \times X \rightarrow X$ given by $\varphi(r, 0) = 0$. In all these cases, the flows only have one trajectory.

Definition 2.5. For a flow X , the $\omega^{\mathbf{r}}$ -limit set (or right-limit set, or positive limit set) of a point $p \in X$ is given as follows:

$$\omega^{\mathbf{r}}(p) = \{q \in X \mid \exists \text{ a net } t_\delta \rightarrow +\infty \text{ such that } t_\delta \cdot p \rightarrow q\}.$$

If \bar{A} denotes the closure of a subset A of a topological space, we note that the subset $\omega^{\mathbf{r}}(p)$ admits the alternative definition

$$\omega^{\mathbf{r}}(p) = \bigcap_{t \geq 0} \overline{[t, +\infty) \cdot p}$$

which has the advantage of showing that $\omega^{\mathbf{r}}(p)$ is closed.

Definition 2.6. The $\Omega^{\mathbf{r}}$ -limit set of a flow X is the following invariant subset:

$$\Omega^{\mathbf{r}}(X) = \bigcup_{p \in X} \omega^{\mathbf{r}}(p)$$

Now we introduce the basic notions of critical, periodic and \mathbf{r} -Poisson stable points.

Definition 2.7. Let X be a flow. A point $x \in X$ is said to be a critical point (or a rest point, or an equilibrium point) if for every $r \in \mathbb{R}$, $r \cdot x = x$. We denote by $C(X)$ the invariant subset of critical points of X .

Definition 2.8. Let X be a flow. A point $x \in X$ is said to be periodic if there is $r \in \mathbb{R}$, $r \neq 0$ such that $r \cdot x = x$. We denote by $P(X)$ the invariant subset of periodic points of X .

It is clear that a critical point is a periodic point. Then

$$C(X) \subset P(X).$$

If $x \in X$ is a periodic point but not critical, then the real $r \neq 0$ such that $r \cdot x = x$ and r is called a *period* of x . The smallest positive period r_0 of x is called the *fundamental period* of x . Further if $r \in \mathbb{R}$ is such that $r \cdot x = x$, then there is $z \in \mathbb{Z}$ such that $r = zr_0$.

Definition 2.9. Let (X, φ) be a flow. A point $x \in X$ is said to be \mathbf{r} -Poisson stable if there is a net $t_\delta \rightarrow +\infty$ such that $t_\delta \cdot x \rightarrow x$; that is, $x \in \omega^{\mathbf{r}}(x)$. We will denote by $P^{\mathbf{r}}(X)$ the invariant subset of \mathbf{r} -Poisson stable points of X .

The reader can easily check that

$$P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X).$$

The notions above can be dualized to obtain the notion of the $\omega^{\mathbf{l}}$ -limit (l for 'left') set of a point p , the $\Omega^{\mathbf{l}}$ -limit of X , l-Poisson stable points, et cetera.

Remark 2.10. *Observe that when X satisfies the first axiom of countability (for instance, when X is metrizable) we can consider sequences instead of nets in definitions 2.5 and 2.9.*

3. END SPACES AND LIMIT SPACES OF AN EXTERIOR SPACE

In this section we will deal with special limit constructions. Observe that if $X = (X, \varepsilon(X))$ is an exterior space, then its externology $\varepsilon(X)$ can be seen as an inverse system of spaces. Composing with different endofunctors $\mathbf{Top} \rightarrow \mathbf{Top}$ and taking the corresponding topological inverse limit we will be able to obtain functors $\mathbf{E} \rightarrow \mathbf{Top}$.

3.1. The functors $L, \tilde{\pi}_0, \check{c}: \mathbf{E} \rightarrow \mathbf{Top}$. Given an exterior space $X = (X, \varepsilon(X))$, its externology $\varepsilon(X)$ is an inverse system of spaces. Then we define the limit space of $(X, \varepsilon(X))$ as the topological space

$$L(X) = \lim \varepsilon(X).$$

Note that for each $E' \in \varepsilon(X)$ the canonical map $\lim \varepsilon(X) \rightarrow E'$ is continuous and factorizes as $\lim \varepsilon(X) \rightarrow \bigcap_{E \in \varepsilon(X)} E \rightarrow E'$. Therefore the canonical map $\lim \varepsilon(X) \rightarrow \bigcap_{E \in \varepsilon(X)} E$ is continuous. On the other side, by the universal property of the inverse system, the family of maps $\bigcap_{E \in \varepsilon(X)} E \rightarrow E', E' \in \varepsilon(X)$ induces a continuous map $\bigcap_{E \in \varepsilon(X)} E \rightarrow \lim \varepsilon(X)$. This implies that the canonical map $\lim \varepsilon(X) \rightarrow \bigcap_{E \in \varepsilon(X)} E$ defines a natural homeomorphism.

We recall that for a topological space Y , $\pi_0(Y)$ denotes the set of path-components of Y and we have a canonical map $Y \rightarrow \pi_0(Y)$ which induces a quotient topology on $\pi_0(Y)$. Similarly, If $c(Y)$ denotes the set on connected components of a space Y , we have a similar quotient map $Y \rightarrow c(Y)$. We remark that if Y is locally path-connected (respectively, locally connected), then $\pi_0(Y)$ ($c(Y)$) is a discrete space.

It is also interesting to note that for any topological space Y , there exists a canonical commutative diagram of natural maps:

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ \pi_0(Y) & \longrightarrow & c(Y) \end{array}$$

Definition 3.1. *Given an exterior space $X = (X, \varepsilon(X))$ the limit space of X is the topological subspace*

$$L(X) = \lim \varepsilon(X) = \bigcap_{E \in \varepsilon(X)} E.$$

The end space of X is the inverse limit

$$\tilde{\pi}_0(X) = \lim \pi_0 \varepsilon(X) = \lim_{E \in \varepsilon(X)} \pi_0(E)$$

provided with the inverse limit topology of the spaces $\pi_0(E)$.

The c-end space of X is the inverse limit

$$\check{c}(X) = \lim c \varepsilon(X) = \lim_{E \in \varepsilon(X)} c(E)$$

provided with the inverse limit topology of the spaces $c(E)$. The elements of $\tilde{\pi}_0(X)$ or $\check{c}(X)$ will be called end points of X .

An end point $a \in \check{\pi}_0(X)$ is represented by the filter base

$$\{U_a^E | U_a^E \text{ is a path-component of } E, E \in \varepsilon(X)\}.$$

We note that a locally path-connected exterior space $(X, \varepsilon(X))$ induces the following family of exterior spaces

$$\{(X, \varepsilon(X, a)) | a \in \check{\pi}_0(X)\}$$

where $\varepsilon(X, a)$ is the externology generated by the filter base

$$\{U_a^E | U_a^E \text{ is a path-component of } E, E \in \varepsilon(X)\}.$$

The end points of $\check{c}(X)$ have similar properties.

It is interesting to observe that if X is an exterior space and X is locally path-connected (locally connected), then $\check{\pi}_0(X)$ ($\check{c}(X)$) is a prodiscrete space. On the other hand, given any exterior space $(X, \varepsilon(X))$, we have a canonical commutative diagram of natural maps

$$\begin{array}{ccc} & L(X) & \\ e_0 \swarrow & & \searrow e \\ \check{\pi}_0(X) & \xrightarrow{\theta} & \check{c}(X) \end{array}$$

Definition 3.2. Given an exterior space $X = (X, \varepsilon(X))$, an end point $a \in \check{\pi}_0(X)$ ($a \in \check{c}(X)$) is said to be e_0 -representable (e -representable) if there is $p \in L(X)$ such that $e_0(p) = a$ ($e(p) = a$). Notice that the maps $e_0: L(X) \rightarrow \check{\pi}_0(X)$, $e: L(X) \rightarrow \check{c}(X)$ induce an e_0 -decomposition and an e -decomposition

$$L(X) = \bigsqcup_{a \in \check{\pi}_0(X)} L_a^0(X), \quad L(X) = \bigsqcup_{a \in \check{c}(X)} L_a(X)$$

where $L_a^0(X) = e_0^{-1}(a)$ and $L_a(X) = e^{-1}(a)$. These special subsets will be respectively called the e_0 -component of the end $a \in \check{\pi}_0(X)$ and the e -component of the end $a \in \check{c}(X)$ in the limit $L(X)$.

We denote by $e_0 L(X)$ and $eL(X)$ the corresponding subsets of representable end points. It is clear that

$$L(X) = \bigsqcup_{a \in e_0 L(X)} L_a^0(X), \quad L(X) = \bigsqcup_{a \in eL(X)} L_a(X)$$

and for $b \in eL(X)$ one has that

$$L_b(X) = \bigsqcup_{a \in (\theta^{-1}(b) \cap e_0 L(X))} L_a^0(X).$$

Example 3.3. Let $M: \mathbb{R} \rightarrow (0, 1)$ be an increasing continuous map such that $\lim_{t \rightarrow -\infty} M(t) = 0$ and $\lim_{t \rightarrow +\infty} M(t) = 1$ and take $A = \{e^{2\pi it} | t \in \mathbb{R}\}$, $B = \{M(t)e^{2\pi it} | t \in \mathbb{R}\}$. Consider $X = A \cup B \subset \mathbb{C}$ provided with the relative topology (observe that X is not locally connected). On the topological space X the flow $\varphi: \mathbb{R} \times X \rightarrow X$ is given by $\varphi(r, e^{2\pi it}) = e^{2\pi i(r+t)}$, $\varphi(r, M(t)e^{2\pi it}) = M(r+t)e^{2\pi i(r+t)}$. It is clear that this flow has two trajectories A, B . If for each natural number n we denote $B_n = \{M(t)e^{2\pi it} | t \geq n\}$, then a base of an externology on X is given by

$\{E_n = A \cup B_n | n \in \mathbb{N}\}$. Since A, B_n are path-connected and $\overline{B_n} = E_n$ is connected, it follows that $\pi_0(E_n) = \{A, B_n\}$ and $c(E_n) = \{E_n\}$. Therefore

$$\tilde{\pi}_0(X) = \{*_A, *_B\}, \quad \check{c}(X) = \{*\}$$

For this example we have $L(X) = A$, the e_0 -decomposition

$$L_{*_A}^0 = A, \quad L_{*_B}^0 = \emptyset$$

and the e -decomposition $L_* = A$. This means that $*_B$ is not e_0 -representable.

It is not difficult to check that the functor L preserves homotopies and the functors $\tilde{\pi}_0, \check{c}$ are invariant by exterior homotopy.

Lemma 3.4. *Suppose that X and Y are exterior spaces and $f, g: X \rightarrow Y$ exterior maps.*

- (i) *If $H: X \bar{\times} I \rightarrow Y$ is an exterior homotopy from f to g , then $L(H) = H|_{L(X) \times I}: L(X \bar{\times} I) = L(X) \times I \rightarrow L(Y)$ is a homotopy from $L(f)$ to $L(g)$.*
- (ii) *If f is exterior homotopic to g , then $\tilde{\pi}_0(f) = \tilde{\pi}_0(g)$ and $\check{c}(f) = \check{c}(g)$.*

As a consequence of this lemma one has:

Proposition 3.5. *The functors $L, \tilde{\pi}_0, \check{c}: \mathbf{E} \rightarrow \mathbf{Top}$ induce functors*

$$L: \pi\mathbf{E} \rightarrow \pi\mathbf{Top}, \quad \tilde{\pi}_0, \check{c}: \pi\mathbf{E} \rightarrow \mathbf{Top}.$$

It is interesting to observe that the functor $L: \mathbf{E} \rightarrow \mathbf{Top}$ admits in a natural way a left adjoint: Given a topological space X , recall that we can consider on X the trivial externology $\varepsilon^{tr}(X) = \{X\}$. This construction gives the exterior space $X_{tr} = (X, \varepsilon^{tr}(X))$ and induces the canonical functor $(\cdot)_{tr}: \mathbf{Top} \rightarrow \mathbf{E}$, $X \mapsto X_{tr}$. The reader can straightforwardly check the following result:

Proposition 3.6. *The functor $(\cdot)_{tr}: \mathbf{Top} \rightarrow \mathbf{E}$ is left adjoint to the functor $L: \mathbf{E} \rightarrow \mathbf{Top}$. Moreover, this pair of adjoint functors induces an adjunction on the homotopy categories: $(\cdot)_{tr}: \pi\mathbf{Top} \rightarrow \pi\mathbf{E}$, $L: \pi\mathbf{E} \rightarrow \pi\mathbf{Top}$.*

3.2. The functors $\bar{L}, \tilde{\pi}_0, \check{c}: \mathbf{E} \rightarrow \mathbf{Top}$. The externology of an exterior space $X = (X, \varepsilon(X))$ and the closure operator of the subjacent topological space induce the following inverse system $\bar{\varepsilon}(X) = \{\bar{E} | E \in \varepsilon(X)\}$. Using this new inverse system, we can rewrite notions and analogous results of subsection above as follows:

Definition 3.7. *Given an exterior space $X = (X, \varepsilon(X))$ the bar-limit space of X is the topological subspace*

$$\bar{L}(X) = \lim \bar{\varepsilon}(X) = \cap_{E \in \varepsilon(X)} \bar{E}.$$

The bar-end space of X is the inverse limit

$$\tilde{\pi}_0(X) = \lim \pi_0 \bar{\varepsilon}(X) = \lim_{E \in \varepsilon(X)} \pi_0(\bar{E})$$

provided with the inverse limit topology of the spaces $\pi_0(\bar{E})$.

The c-bar-end space of X is the inverse limit

$$\check{c}(X) = \lim c \bar{\varepsilon}(X) = \lim_{E \in \varepsilon(X)} c(\bar{E})$$

provided with the inverse limit topology of the spaces $c(\bar{E})$.

Given any exterior space $X = (X, \varepsilon(X))$, we have a canonical diagram of natural maps

$$\begin{array}{ccc} & \bar{L}(X) & \\ \bar{e}_0 \swarrow & & \searrow \bar{e} \\ \check{\pi}_0(X) & \xrightarrow{\bar{\theta}} & \check{c}(X) \end{array}$$

and there canonical natural maps $L(X) \subset \overline{L(X)} \subset \bar{L}(X)$, $\check{\pi}_0(X) \rightarrow \check{\pi}_0(X)$, $\check{c}(X) \rightarrow \check{c}(X)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & L(X) & & & \\ & \swarrow \downarrow \searrow & & & \\ \check{\pi}_0(X) & \xrightarrow{\quad} & \check{c}(X) & & \\ \downarrow & & \downarrow & & \\ & \bar{L}(X) & & & \\ & \swarrow \downarrow \searrow & & & \\ \check{\pi}_0(X) & \xrightarrow{\quad} & \check{c}(X) & & \end{array}$$

Definition 3.8. Given an exterior space $X = (X, \varepsilon(X))$, an end point $a \in \check{\pi}_0(X)$ ($a \in \check{c}(X)$) is said to be \bar{e}_0 -representable (\bar{e} -representable) if there is $p \in \bar{L}(X)$ such that $\bar{e}_0(p) = a$ ($\bar{e}(p) = a$). The maps $\bar{e}_0: \bar{L}(X) \rightarrow \check{\pi}_0(X)$, $\bar{e}: \bar{L}(X) \rightarrow \check{c}(X)$ induce an \bar{e}_0 -decomposition and an \bar{e} -decomposition

$$\bar{L}(X) = \bigsqcup_{a \in \check{\pi}_0(X)} \bar{L}_a^0(X), \quad \bar{L}(X) = \bigsqcup_{a \in \check{c}(X)} \bar{L}_a(X)$$

where $\bar{L}_a^0(X) = \bar{e}_0^{-1}(a)$ ($\bar{L}_a(X) = \bar{e}^{-1}(a)$) will be called the \bar{e}_0 -component (\bar{e} -component) of the end $a \in \check{\pi}_0(X)$ ($a \in \check{c}(X)$) in the limit $\bar{L}(X)$.

We denote by $\bar{e}_0 \bar{L}(X)$ ($\bar{e} \bar{L}(X)$) the corresponding subset of representable end points. It is clear that we have a commutative diagram

$$\begin{array}{ccccc} & L(X) & & & \\ & \swarrow \downarrow \searrow & & & \\ e_0 L(X) & \xrightarrow{\quad} & e L(X) & & \\ \downarrow & & \downarrow & & \\ & \bar{L}(X) & & & \\ & \swarrow \downarrow \searrow & & & \\ \bar{e}_0 \bar{L}(X) & \xrightarrow{\quad} & \bar{e} \bar{L}(X) & & \end{array}$$

We also have the following similar results:

Lemma 3.9. Suppose that X, Y are exterior spaces and $f, g: X \rightarrow Y$ exterior maps.

- (i) If $H: X \bar{\times} I \rightarrow Y$ is an exterior homotopy from f to g , then $\bar{L}(H) = H|_{\bar{L}(X) \times I}: \bar{L}(X \bar{\times} I) = \bar{L}(X) \times I \rightarrow \bar{L}(Y)$ is a homotopy from $\bar{L}(f)$ to $\bar{L}(g)$.
- (ii) If f is exterior homotopic to g , then $\tilde{\pi}_0(f) = \tilde{\pi}_0(g)$ and $\check{c}(f) = \check{c}(g)$.

As a consequence of this lemma we have:

Proposition 3.10. *The functors $\bar{L}, \tilde{\pi}_0, \check{c}: \mathbf{E} \rightarrow \mathbf{Top}$ induce functors*

$$\bar{L}: \pi \mathbf{E} \rightarrow \pi \mathbf{Top}, \quad \tilde{\pi}_0, \check{c}: \pi \mathbf{E} \rightarrow \mathbf{Top}.$$

In Proposition 3.6 a left adjoint has been constructed for the functor L . Nevertheless in the case of functor \bar{L} we have the following alternative result:

Proposition 3.11. *Suppose that X, Y are exterior spaces and X satisfies that for every $E \in \varepsilon(X)$, $\bar{E} = X$. Then we have the following canonical injective map*

$$Hom_{\mathbf{E}}(X, Y) \rightarrow Hom_{\mathbf{Top}}(X_t, \bar{L}(Y)),$$

where X_t denote the subjacent topological space of X . Therefore, if \mathbf{E}_{den} denote the full subcategory of exterior spaces X satisfying that for every $E \in \varepsilon(X)$, $\bar{E} = X$, then $\bar{L}: \mathbf{E}_{\text{den}} \rightarrow \mathbf{Top}$ is a faithful functor.

3.3. Topological and exterior properties and canonical maps. Let $X = (X, \varepsilon(X))$ be an exterior space and consider $\bar{\varepsilon}(X) = \{\bar{E} | E \in \varepsilon(X)\}$.

Definition 3.12. *An exterior space $X = (X, \varepsilon(X))$ is said to be regular at infinity (respectively, locally compact at infinity) if for every $E \in \varepsilon(X)$, there exists $E' \in \varepsilon(X)$ such that $\bar{E}' \subset E$ (\bar{E}' is compact and $\bar{E}' \subset E$).*

Obviously, locally compact at infinity implies regular at infinity.

Example 3.13. *As an example of regular at infinity exterior space we can take any Hausdorff locally compact space X provided with its cocompact externology. Now, if K is a compact subset of X and we take the externology of open neighborhoods of K in X , we obtain a new exterior space which is locally compact at infinity. Next, we describe an exterior space which is not regular at infinity: Consider the following planar differential system*

$$\frac{d\alpha}{dt} = f(\alpha, \beta), \quad \frac{d\beta}{dt} = af(\alpha, \beta)$$

where $f(\alpha, \beta) > 0$, $f(\alpha + 1, \beta) = f(\alpha + 1, \beta + 1) = f(\alpha, \beta + 1)$ and a is an irrational fixed number. This system induces a flow on the torus $X = S^1 \times S^1$, which satisfies that each trajectory is dense in X . Take the externology $\varepsilon(X)$ constituted by those open subsets E such that for any $p \in X$ there is $r_0 \in \mathbb{R}$ such that $r \cdot p \in E$, for all $r \geq r_0$. Then it is easy to check that the exterior space $(X, \varepsilon(X))$ is not regular at infinity.

Proposition 3.14. *If an exterior space $X = (X, \varepsilon(X))$ is regular at infinity, then*

- (i) $L(X) = \bar{L}(X)$
- (ii) $\tilde{\pi}_0(X) = \tilde{\pi}_0(X)$, $e_0 L(X) = \bar{e}_0 \bar{L}(X)$
- (iii) $\check{c}(X) = \check{c}(X)$, $eL(X) = \bar{e} \bar{L}(X)$.

Proposition 3.15. *Suppose that $X = (X, \varepsilon(X))$ is an exterior space.*

- (i) *If X is locally path-connected, then $\tilde{\pi}_0(X) = \check{c}(X)$, $e_0 L(X) = eL(X)$,*
- (ii) *If X is locally path-connected and regular at infinity, then $\tilde{\pi}_0(X) = \tilde{\pi}_0(X) = \check{c}(X) = \check{c}(X)$, $e_0 L(X) = \bar{e}_0 \bar{L}(X) = eL(X) = \bar{e} \bar{L}(X)$.*

Proposition 3.16. *If an exterior space $X = (X, \varepsilon(X))$ is locally compact at infinity, then $L(X) = \bar{L}(X)$ is compact.*

Proof. Since X is regular at infinity, by Proposition 3.14, $L(X) = \bar{L}(X)$. Take $E_0 \in \varepsilon(X)$ such that $\overline{E_0}$ is compact, then the closed subset satisfies $\bar{L}(X) \subset \overline{E_0}$. Therefore $L(X) = \bar{L}(X)$ is compact. \square

Theorem 3.17. *Let $X = (X, \varepsilon(X))$ be an exterior space and suppose that X is locally path-connected and locally compact at infinity. Then,*

- (i) $L(X) = \bar{L}(X)$ is compact,
- (ii) $e_0 L(X) = \bar{e}_0 \bar{L}(X) = eL(X) = \bar{e} \bar{L}(X) = \tilde{\pi}_0(X) = \check{\pi}_0(X) = \check{c}(X) = \check{\bar{c}}(X)$ (any end point is representable by a point of the limit),
- (iii) $\tilde{\pi}_0(X) = \check{\pi}_0(X) = \check{c}(X) = \check{\bar{c}}(X)$ is a profinite compact space,
- (iv) If $a \in \tilde{\pi}_0(X) = \check{\pi}_0(X) = \check{c}(X) = \check{\bar{c}}(X)$, then $L_a(X) = \bar{L}_a(X) = L_a^0(X) = \bar{L}_a^0(X)$ is a non-empty continuum.

Proof. As a consequence of Propositions 3.15 and 3.16, it follows (i), $\tilde{\pi}_0(X) = \check{\pi}_0(X) = \check{c}(X) = \check{\bar{c}}(X)$ and $e_0 L(X) = \bar{e}_0 \bar{L}(X) = eL(X) = \bar{e} \bar{L}(X)$.

Now take $E_0 \in \varepsilon(X)$ such that $\overline{E_0}$ is compact, then $\bar{\varepsilon}'(X) = \{\bar{E} \mid E \in \varepsilon(X), \bar{E} \subset \overline{E_0}\}$ is cofinal and we have $\bar{L}(X) = \bigcap_{\bar{E} \in \bar{\varepsilon}'(X)} \bar{E}$ and $\check{c}(X) = \lim_{\bar{E} \in \bar{\varepsilon}'(X)} c(\bar{E})$.

Note that any end a can be represented by $\{F\}_{F \in c(\bar{E}), \bar{E} \in \bar{\varepsilon}'(X)}$. Since F is a non-empty component of $\bar{E} \subset \overline{E_0}$, it follows that F is closed (F is a continuum). We also have that the family of closed subset $\{F\}_{F \in c(\bar{E}), \bar{E} \in \bar{\varepsilon}'(X)}$ satisfies the finite intersection property. Since $\overline{E_0}$ is compact, one has that $L_a(X) = \bigcap_{F \in c(\bar{E}), \bar{E} \in \bar{\varepsilon}'(X)} F$ is a non-empty continuum (see Theorem 6.1.20 in [5]). Therefore this end is representable by points of the limit space. Since the map $e: L(X) \rightarrow \tilde{\pi}_0(X)$ is continuous and $L(X)$ is compact it follows that $\tilde{\pi}_0(X)$ is compact. Moreover, since $\tilde{\pi}_0(X) \cong \lim \pi_0(E)$ is prodiscrete, taking into account that $\tilde{\pi}_0(X)$ is compact, we have that $\tilde{\pi}_0(X)$ is a profinite compact space. \square

Remark 3.18. *For an ANR exterior space X , under some topological conditions the shape of the limit space is determined by the resolution $\varepsilon(X)$. Some applications of shape theory to dynamical systems can be seen in [18, 20].*

4. THE CATEGORY OF \mathbf{r} -EXTERIOR FLOWS

We are going to consider the exterior space $\mathbb{R}^{\mathbf{r}} = (\mathbb{R}, \mathbf{r})$, where \mathbf{r} is the following externology:

$$\mathbf{r} = \{U \mid U \text{ is open and there is } n \in \mathbb{N} \text{ such that } (n, +\infty) \subset U\}.$$

Note that a base for \mathbf{r} is given by $\mathcal{B}(\mathbf{r}) = \{(n, +\infty) \mid n \in \mathbb{N}\}$.

The exterior space $\mathbb{R}^{\mathbf{r}}$ plays an important role in the definition of \mathbf{r} -exterior flow below. Such notion mixes the structures of dynamical system and exterior space:

Definition 4.1. *Let M be an exterior space, M_t denote the subjacent topological space and M_d denote the set M provided with the discrete topology. An \mathbf{r} -exterior flow is a continuous flow $\varphi: \mathbb{R} \times M_t \rightarrow M_t$ such that $\varphi: \mathbb{R} \times M_d \rightarrow M$ is exterior and for any $t \in \mathbb{R}$, $F_t: M \times I \rightarrow M$, $F_t(x, s) = \varphi(ts, x)$, $s \in I$, $x \in M$, is also exterior.*

An \mathbf{r} -exterior flow morphism of \mathbf{r} -exterior flows $f: M \rightarrow N$ is a flow morphism such that f is exterior. We will denote by $\mathbf{E}^{\mathbf{r}}\mathbf{F}$ the category of \mathbf{r} -exterior flows and \mathbf{r} -exterior flow morphisms.

Given an \mathbf{r} -exterior flow $(M, \varphi) \in \mathbf{E}^{\mathbf{r}}\mathbf{F}$, one also has a flow $(M_{\mathbf{t}}, \varphi) \in \mathbf{F}$. This gives a forgetful functor

$$(\cdot)_{\mathbf{t}}: \mathbf{E}^{\mathbf{r}}\mathbf{F} \rightarrow \mathbf{F}.$$

Now given a continuous flow $X = (X, \varphi)$, an open $N \in \mathbf{t}_X$ is said to be \mathbf{r} -exterior if for any $x \in X$ there is $T^x \in \mathbf{r}$ such that $\varphi(T^x \times \{x\}) \subset N$. It is easy to check that the family of \mathbf{r} -exterior subsets of X is an externology, denoted by $\varepsilon^{\mathbf{r}}(X)$, which gives an exterior space $X^{\mathbf{r}} = (X, \varepsilon^{\mathbf{r}}(X))$ such that $\varphi: \mathbb{R}^{\mathbf{r}} \times X_{\mathbf{d}} \rightarrow X^{\mathbf{r}}$ is exterior and $F_t: X^{\mathbf{r}} \times I \rightarrow X^{\mathbf{r}}$, $F_t(x, s) = \varphi(ts, x)$, is also exterior for every $t \in \mathbb{R}$. Therefore $(X^{\mathbf{r}}, \varphi)$ is an \mathbf{r} -exterior flow which is said to be the \mathbf{r} -exterior flow associated to X . When there is no possibility of confusion, $(X^{\mathbf{r}}, \varphi)$ will be briefly denoted by $X^{\mathbf{r}}$. Then we have a functor

$$(\cdot)^{\mathbf{r}}: \mathbf{F} \rightarrow \mathbf{E}^{\mathbf{r}}\mathbf{F}.$$

Note that for a flow (X, φ) , if E is an open subset such that \overline{E} is compact, then E is an \mathbf{r} -exterior subset if and only if \overline{E} is an “absorbing region” in the sense of Definition 1.4.2 in [1]. On the other hand, the forgetful functor and the given constructions of exterior flows are related as follows:

Proposition 4.2. *The functor $(\cdot)^{\mathbf{r}}: \mathbf{F} \rightarrow \mathbf{E}^{\mathbf{r}}\mathbf{F}$ is left adjoint to the functor $(\cdot)_{\mathbf{t}}: \mathbf{E}^{\mathbf{r}}\mathbf{F} \rightarrow \mathbf{F}$. Moreover $(\cdot)_{\mathbf{t}}(\cdot)^{\mathbf{r}} = \text{id}$ and \mathbf{F} can be considered as a full subcategory of $\mathbf{E}^{\mathbf{r}}\mathbf{F}$ via $(\cdot)^{\mathbf{r}}$.*

Proof. Let X be in \mathbf{F} and M be in $\mathbf{E}^{\mathbf{r}}\mathbf{F}$. If $f: X^{\mathbf{r}} \rightarrow M$ is a morphism in $\mathbf{E}^{\mathbf{r}}\mathbf{F}$, then it is clear that $f: X = (X^{\mathbf{r}})_{\mathbf{t}} \rightarrow M_{\mathbf{t}}$ is a morphism in \mathbf{F} . Now take $g: X \rightarrow M_{\mathbf{t}}$ a morphism in \mathbf{F} and $E \in \varepsilon(M)$. Given any $x \in X$ one has $g(x) \in M$ and, taking into account that M is an \mathbf{r} -exterior flow, there exists $T^{g(x)}$ such that $T^{g(x)} \cdot g(x) \subset E$. This implies that $T^{g(x)} \cdot x \subset g^{-1}(E)$. Therefore $g^{-1}(E) \in \varepsilon(X^{\mathbf{r}}) = \varepsilon^{\mathbf{r}}(X)$. \square

4.1. End Spaces and Limit Spaces of an exterior flow. In section 3 we have defined the end and limit spaces of an exterior space. In particular, since any \mathbf{r} -exterior flow X is an exterior space, we can consider the end spaces $\tilde{\pi}_0(X)$, $\check{c}(X)$, $\tilde{\pi}_0(X)$, $\check{c}(X)$ and the limit spaces $L(X)$, $\bar{L}(X)$. Notice that one has the following properties:

Proposition 4.3. *Suppose that $X = (X, \varphi)$ is an \mathbf{r} -exterior flow. Then*

- (i) *The spaces $L(X)$, $\bar{L}(X)$ are invariant;*
- (ii) *There are trivial flows induced on $\tilde{\pi}_0(X)$, $\check{c}(X)$, $\tilde{\pi}_0(X)$ and $\check{c}(X)$.*

Proof. (i): We have that $L(X) = \cap_{E \in \varepsilon(X)} E$. Note that for any $s \in \mathbb{R}$, $\varphi_s(E) \in \varepsilon(X)$ if and only if $E \in \varepsilon(X)$. Then $\varphi_s(L(X)) = \varphi_s(\cap_{E \in \varepsilon(X)} E) = \cap_{E \in \varepsilon(X)} \varphi_s(E) = \cap_{E \in \varepsilon(X)} E = L(X)$. In a similar way, it can be checked that $\bar{L}(X)$ is also invariant.

(ii): For any $s \in \mathbb{R}$, consider the exterior homotopy $F_s: X \times I \rightarrow X$, $F_s(x, t) = \varphi(ts, x)$, from id_X to φ_s . By Lemma 3.4, it follows that $\text{id} = \tilde{\pi}_0(\varphi_s)$. Therefore the induced action is trivial. In the other cases the proof is similar using Lemma 3.9. \square

As a consequence of this result, one has functors $L, \tilde{\pi}_0, \check{c}, \bar{L}, \tilde{\pi}_0, \check{c}: \mathbf{E}^{\mathbf{r}}\mathbf{F} \rightarrow \mathbf{F}$.

Proposition 4.4. *The functors $L, \tilde{\pi}_0, \check{c}, \bar{L}, \check{\pi}_0, \check{c}: \mathbf{E}^{\mathbf{r}}\mathbf{F} \rightarrow \mathbf{F}$ induce functors*

$$L, \bar{L}: \pi\mathbf{E}^{\mathbf{r}}\mathbf{F} \rightarrow \pi\mathbf{F}, \quad \tilde{\pi}_0, \check{c}, \check{\pi}_0, \check{c}: \pi\mathbf{E}^{\mathbf{r}}\mathbf{F} \rightarrow \mathbf{F},$$

where the homotopy categories are constructed in a canonical way.

4.2. The end point of a trajectory and the induced decompositions of an exterior flow. For a \mathbf{r} exterior flow X , one has that each trajectory has an end point given as follows: Given $p \in X$ and $E \in \varepsilon(X)$, there is $T^p \in \mathbf{r}$ such that $T^p \cdot p \subset E$. We can suppose that T^p is path-connected and therefore so is $T^p \cdot p$; this way there is a unique path-component $\omega_{\mathbf{r}}^0(p, E)$ (component $\omega_{\mathbf{r}}(p, E)$) of E such that $T^p \cdot p \subset \omega_{\mathbf{r}}^0(p, E) \subset E$ ($T^p \cdot p \subset \omega_{\mathbf{r}}(p, E) \subset E$). This gives set maps $\omega_{\mathbf{r}}^0(\cdot, E): X \rightarrow \pi_0(E)$ and $\omega_{\mathbf{r}}^0: X \rightarrow \tilde{\pi}_0(X)$ ($\omega_{\mathbf{r}}(\cdot, E): X \rightarrow c(E)$ and $\omega_{\mathbf{r}}: X \rightarrow \check{c}(X)$) such that the following diagram commutes:

$$\begin{array}{ccc} L(X) & & \\ \swarrow e_0 & \downarrow & \searrow e \\ & X & \\ \swarrow \omega_{\mathbf{r}}^0 & & \searrow \omega_{\mathbf{r}} \\ \tilde{\pi}_0(X) & \xrightarrow{\theta} & \check{c}(X) \end{array}$$

These maps permit to divide a flow in simpler flows.

Definition 4.5. *Let X be an \mathbf{r} -exterior flow. We will consider $X_{(\mathbf{r},a)}^0 = (\omega_{\mathbf{r}}^0)^{-1}(a)$, $a \in \tilde{\pi}_0(X)$ and $X_{(\mathbf{r},b)} = \omega_{\mathbf{r}}^{-1}(b)$, $b \in \check{c}(X)$. The invariant spaces $X_{(\mathbf{r},a)}^0$ and $X_{(\mathbf{r},b)}$ will be called the $\omega_{\mathbf{r}}^0$ -basin at $a \in \tilde{\pi}_0(X)$ and the $\omega_{\mathbf{r}}$ -basin at $b \in \check{c}(X)$, respectively.*

The maps $\omega_{\mathbf{r}}^0$ and $\omega_{\mathbf{r}}$ induce the following partitions of X in simpler flows

$$X = \bigsqcup_{a \in \tilde{\pi}_0(X)} X_{(\mathbf{r},a)}^0, \quad X = \bigsqcup_{b \in \check{c}(X)} X_{(\mathbf{r},b)}$$

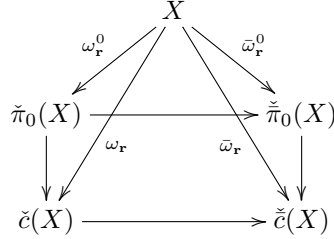
that will be called respectively, the $\omega_{\mathbf{r}}^0$ -decomposition and the $\omega_{\mathbf{r}}$ -decomposition of the \mathbf{r} -exterior flow X .

Similarly, given $p \in X$, if $\omega_{\mathbf{r}}^0(p, E)$ is the path-component of E such that $T^p \cdot p \subset \omega_{\mathbf{r}}^0(p, E) \subset E$, then we also have that $T^p \cdot p \subset \omega_{\mathbf{r}}^0(p, E) \subset \bar{\omega}_{\mathbf{r}}^0(p, \bar{E}) \subset \bar{E}$, where $\bar{\omega}_{\mathbf{r}}^0(p, \bar{E})$ is the unique path-component of \bar{E} containing $T^p \cdot p$. In the same way as above, we have maps $\bar{\omega}_{\mathbf{r}}^0(\cdot, \bar{E}): X \rightarrow \pi_0(\bar{E})$ and $\bar{\omega}_{\mathbf{r}}^0: X \rightarrow \check{\pi}_0(X)$. Analogously, we obtain set maps $\bar{\omega}_{\mathbf{r}}(\cdot, \bar{E}): X \rightarrow c(\bar{E})$ and $\bar{\omega}_{\mathbf{r}}: X \rightarrow \check{c}(X)$ such that the following diagram commutes:

$$\begin{array}{ccc} \bar{L}(X) & & \\ \swarrow \bar{e}_0 & \downarrow & \searrow \bar{e} \\ & X & \\ \swarrow \bar{\omega}_{\mathbf{r}}^0 & & \searrow \bar{\omega}_{\mathbf{r}} \\ \check{\pi}_0(X) & \xrightarrow{\bar{\theta}} & \check{c}(X) \end{array}$$

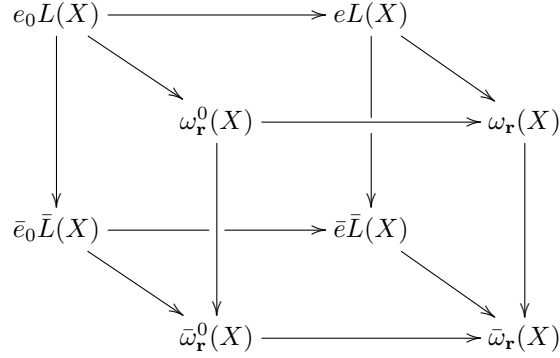
Remark 4.6. *It is important to note that the maps $\omega_{\mathbf{r}}^0$, $\omega_{\mathbf{r}}$, $\bar{\omega}_{\mathbf{r}}^0$, $\bar{\omega}_{\mathbf{r}}$ need not be continuous.*

As in the case above we can consider the corresponding $\bar{\omega}_r^0$ -basin and $\bar{\omega}_r$ -basin, denoted by $\bar{X}_{(r,a)}^0 = (\bar{\omega}_r^0)^{-1}(a)$, $\bar{X}_{(r,a)} = \bar{\omega}_r^{-1}(a)$, respectively, and their induced decompositions. We also note that the following diagram commutes:



Definition 4.7. Let X be an r -exterior flow. An end point $a \in \tilde{\pi}_0(X)$ is said to be ω_r^0 -representable (similarly for $\omega_r, \bar{\omega}_r^0, \bar{\omega}_r$) if there is $p \in X$ such that $\omega_r^0(p) = a$.

Denote by $\omega_r^0(X)$ the space of ω_r^0 -representable end points (similarly, $\omega_r(X), \bar{\omega}_r^0(X), \bar{\omega}_r(X)$). Since the ω -decompositions of X are compatible with the e -decompositions of the limit subspace, we have the following commutative diagram of representable end points:



Proposition 4.8. Let X be an r -exterior flow. Then

- (i) $\omega^r(p) \subset \bar{L}_{\bar{\omega}_r(p)}(X)$, for any $p \in X$.
- (ii) If $a \in \check{c}(X)$ is $\bar{\omega}_r$ -representable (that is, $\bar{X}_{(r,a)} = \bar{\omega}_r^{-1}(a) \neq \emptyset$) and there exists $p \in \bar{X}_{(r,a)}$ such that $\omega^r(p) \neq \emptyset$, then a is \bar{e} -representable.

Proof. (i) If $q \in \omega^r(p)$, then $q \in \cap_{T \in r} \overline{T \cdot p}$. On the other hand, given $E \in \varepsilon(X)$, if $\bar{\omega}_r(a, \bar{E})$ is the connected component of \bar{E} determined by a , since $\bar{\omega}_r(p) = a$, there is $T \in r$ such that $\overline{T \cdot p} \subset \bar{\omega}_r(a, \bar{E})$. Then $q \in \bar{\omega}_r(a, \bar{E})$ for every $E \in \varepsilon(X)$ and $q \in \bar{L}(X)$. This implies that $\bar{e}(q) = a$ for any $q \in \omega^r(p)$. Therefore $\omega^r(p) \subset \bar{L}_{\bar{\omega}_r(p)}(X)$.
(ii) It follows from (i). \square

Proposition 4.9. Let X be an r -exterior flow and for $p \in X$ denote $\gamma_r(p) = \{r \cdot p \mid r \geq 0\}$. If $\gamma_r(p) \cap \bar{L}(X) \neq \emptyset$, then $\bar{\omega}_r(p)$ is \bar{e} -representable.

Proof. We note that $\overline{\gamma_r(p)} = \gamma_r(p) \cup \omega^r(p)$. If $p \in \bar{L}(X)$, then $\bar{\omega}_r(p) = \bar{e}(p)$ and $\bar{\omega}_r(p)$ is \bar{e} -representable. If $p \notin \bar{L}(X)$, then $\gamma_r(p) \cap \bar{L}(X) = \emptyset$ and $\overline{\gamma_r(p)} \cap \bar{L}(X) = \omega^r(p)$. Then $\omega^r(p) \neq \emptyset$ and, by Proposition 4.8 above, $\bar{\omega}_r(p)$ is \bar{e} -representable. \square

5. END AND LIMIT SPACES OF A FLOW VIA EXTERIOR FLOWS

Recall that we have considered the functor:

$$(\cdot)^{\mathbf{r}}: \mathbf{F} \rightarrow \mathbf{E}^{\mathbf{r}}\mathbf{F}$$

and the functors:

$$L, \tilde{\pi}_0, \check{c}, \bar{L}, \tilde{\pi}_0^{\mathbf{r}}, \check{c}^{\mathbf{r}}: \mathbf{E}^{\mathbf{r}}\mathbf{F} \rightarrow \mathbf{F}.$$

Therefore we can consider the composites:

$$L^{\mathbf{r}} = L(\cdot)^{\mathbf{r}}, \tilde{\pi}_0^{\mathbf{r}} = \tilde{\pi}_0(\cdot)^{\mathbf{r}}, \check{c}^{\mathbf{r}} = \check{c}(\cdot)^{\mathbf{r}}, \bar{L}^{\mathbf{r}} = \bar{L}(\cdot)^{\mathbf{r}}, \tilde{\pi}_0^{\mathbf{r}} = \tilde{\pi}_0(\cdot)^{\mathbf{r}}, \check{c}^{\mathbf{r}} = \check{c}(\cdot)^{\mathbf{r}}$$

to obtain functors $L^{\mathbf{r}}, \tilde{\pi}_0^{\mathbf{r}}, \check{c}^{\mathbf{r}}, \bar{L}^{\mathbf{r}}, \tilde{\pi}_0^{\mathbf{r}}, \check{c}^{\mathbf{r}}: \mathbf{F} \rightarrow \mathbf{F}$.

In this way, given a flow X , we have the end spaces $\tilde{\pi}_0^{\mathbf{r}}(X) = \tilde{\pi}_0(X^{\mathbf{r}})$, $\check{c}^{\mathbf{r}}(X) = \check{c}(X^{\mathbf{r}})$, the limit space $L^{\mathbf{r}}(X) = L(X^{\mathbf{r}})$, the bar-end spaces $\tilde{\pi}_0^{\mathbf{r}}(X) = \tilde{\pi}_0(X^{\mathbf{r}})$, $\check{c}^{\mathbf{r}}(X) = \check{c}(X^{\mathbf{r}})$ and the bar-limit space $\bar{L}^{\mathbf{r}}(X) = \bar{L}(X^{\mathbf{r}})$.

Similarly, using the associated exterior flow $X^{\mathbf{r}}$, we denote

$$X_{(\mathbf{r},a)}^0 = (\omega_{\mathbf{r}}^0)^{-1}(a), \quad a \in \tilde{\pi}_0^{\mathbf{r}}(X)$$

$$X_{(\mathbf{r},a)} = \omega_{\mathbf{r}}^{-1}(a), \quad a \in \check{c}^{\mathbf{r}}(X)$$

The maps $\omega_{\mathbf{r}}^0, \omega_{\mathbf{r}}$ induce the following partitions of X in simpler flows

$$X = \bigsqcup_{a \in \tilde{\pi}_0^{\mathbf{r}}(X)} X_{(\mathbf{r},a)}^0, \quad X = \bigsqcup_{a \in \check{c}^{\mathbf{r}}(X)} X_{(\mathbf{r},a)}$$

that will be called respectively, the $\omega_{\mathbf{r}}^0$ -decomposition and the $\omega_{\mathbf{r}}$ -decomposition of the flow X . Now take

$$\bar{X}_{(\mathbf{r},a)}^0 = (\bar{\omega}_{\mathbf{r}}^0)^{-1}(a), \quad a \in \tilde{\pi}_0^{\mathbf{r}}(X)$$

$$\bar{X}_{(\mathbf{r},a)} = \bar{\omega}_{\mathbf{r}}^{-1}(a), \quad a \in \check{c}^{\mathbf{r}}(X)$$

the maps $\bar{\omega}_{\mathbf{r}}^0, \bar{\omega}_{\mathbf{r}}$ induce the $\bar{\omega}_{\mathbf{r}}^0$ -decomposition and the $\bar{\omega}_{\mathbf{r}}$ -decomposition of the flow X :

$$X = \bigsqcup_{a \in \tilde{\pi}_0^{\mathbf{r}}(X)} \bar{X}_{(\mathbf{r},a)}^0, \quad X = \bigsqcup_{a \in \check{c}^{\mathbf{r}}(X)} \bar{X}_{(\mathbf{r},a)}.$$

It is interesting to consider the following equivalence of categories: Given any flow $\varphi: \mathbb{R} \times X \rightarrow X$, one can consider the *reversed flow* $\varphi': \mathbb{R} \times X \rightarrow X$ defined by $\varphi'(r, x) = \varphi(-r, x)$, for every $(r, x) \in \mathbb{R} \times X$. The correspondence, $(X, \varphi) \rightarrow (X, \varphi')$, gives rise to a functor

$$(\cdot)': \mathbf{F} \rightarrow \mathbf{F}$$

which is an equivalence of categories and verifies $(\cdot)'(\cdot)' = \text{id}$. Using the composites

$$L^1 = (\cdot)'L^{\mathbf{r}}(\cdot)', \tilde{\pi}_0^1 = (\cdot)'\tilde{\pi}_0^{\mathbf{r}}(\cdot)', \check{c}^1 = (\cdot)'\check{c}^{\mathbf{r}}(\cdot)'$$

$$\bar{L}^1 = (\cdot)'\bar{L}^{\mathbf{r}}(\cdot)', \tilde{\pi}_0^1 = (\cdot)'\tilde{\pi}_0^{\mathbf{r}}(\cdot)', \check{c}^1 = (\cdot)'\check{c}^{\mathbf{r}}(\cdot)'$$

we obtain the new functors $L^1, \tilde{\pi}_0^1, \check{c}^1, \bar{L}^1, \tilde{\pi}_0^1, \check{c}^1: \mathbf{F} \rightarrow \mathbf{F}$ and the decompositions

$$X = \bigsqcup_{a \in \tilde{\pi}_0^1(X)} X_{(1,a)}^0, \quad X = \bigsqcup_{a \in \check{c}^1(X)} X_{(1,a)}$$

$$X = \bigsqcup_{a \in \tilde{\pi}_0^1(X)} \bar{X}_{(1,a)}^0, \quad X = \bigsqcup_{a \in \check{c}^1(X)} \bar{X}_{(1,a)}.$$

Remark 5.1. All decompositions above can be considered as generalizations for a continuous flow of the disjoint union of “stable” (or “unstable” for the dual case) submanifolds of a differentiable flow(see [17], [18], [19]).

We note that the decompositions of a flow X are compatible with decompositions of limit subspaces.

For a Morse function [16] $f: M \rightarrow \mathbb{R}$, where M is a compact T_2 Riemannian manifold, one has that the opposite of the gradient of the f induces a flow with a finite number of critical points. In this case, we have that M is locally path-connected and the flow is \mathbf{r} -regular at infinity. Then we have all the properties obtained by Theorem 3.17. For instance we can take the height function of a 2-torus:

Example 5.2. Let $\varphi: \mathbb{R} \times (S^1 \times S^1) \rightarrow S^1 \times S^1$ be the flow induced by the opposite of the gradient of the height function with four critical points:

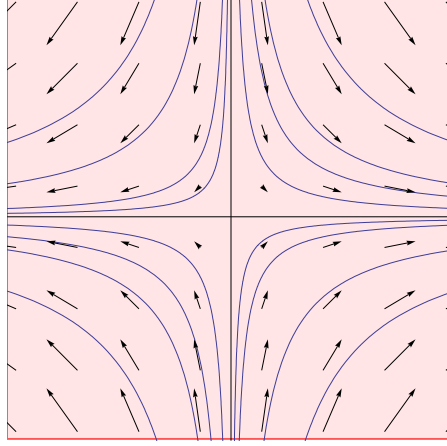


In this example, the limit space, end space and decomposition of the flow and the reverse flow are given in the following table:

$L^{\mathbf{r}} = \{P_0, P_1, P_2, P_3\}$	$\tilde{\pi}_0^{\mathbf{r}} = \{P_0, P_1, P_2, P_3\}$
$L_{P_i}^{\mathbf{r}} = \{P_i\}$	
$X_{(\mathbf{r}, P_3)} = \{P_3\}$	$X_{(\mathbf{r}, P_2)} = \{P_2\} \cup \gamma_2^3 \cup \tilde{\gamma}_2^3$
$X_{(\mathbf{r}, P_1)} = \{P_1\} \cup \gamma_1^2 \cup \tilde{\gamma}_1^2$	$X_{(\mathbf{r}, P_0)} = (S^1 \times S^1) \setminus \bigcup_{i=1}^3 X_{(\mathbf{r}, P_i)}$
$L^{\mathbf{l}} = \{P_0, P_1, P_2, P_3\}$	$\tilde{\pi}_0^{\mathbf{l}} = \{P_0, P_1, P_2, P_3\}$
$L_{P_i}^{\mathbf{l}} = \{P_i\}$	
$X_{(\mathbf{l}, P_0)} = \{P_0\}$	$X_{(\mathbf{l}, P_1)} = \{P_1\} \cup \gamma_0^1 \cup \tilde{\gamma}_0^1$
$X_{(\mathbf{l}, P_2)} = \{P_2\} \cup \gamma_1^2 \cup \tilde{\gamma}_1^2$	$X_{(\mathbf{l}, P_3)} = (S^1 \times S^1) \setminus \bigcup_{i=0}^2 X_{(\mathbf{l}, P_i)}$

Now we consider a flow induced by a lineal differential equation on \mathbb{R}^2 that also induces a new flow on the Alexandrov one-point compactification $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

Example 5.3. Consider on S^2 the flow induced by $\varphi(r, (u_1, u_2)) = (e^{r\lambda_1}u_1, e^{r\lambda_2}u_2)$, $u_1, u_2 \in \mathbb{R}$, $\varphi(r, \infty) = \infty$, $(\lambda_1 > 0, \lambda_2 < 0)$



The limit spaces, end spaces and decomposition of the flow (S^2, ϕ) (as well as the reverse flow) are given in the following table:

$L^{\mathbf{r}} = \{0, \infty\}$	$\tilde{\pi}_0^{\mathbf{r}} = \{0, \infty\}$
$L_0^{\mathbf{r}} = \{0\}$	$L_{\infty}^{\mathbf{r}} = \{\infty\}$
$X_{(\mathbf{r}, 0)} = \{0\} \times \mathbb{R}$	$X_{(\mathbf{r}, \infty)} = ((\mathbb{R} \setminus \{0\}) \times \mathbb{R}) \cup \{\infty\}$
$L^{\mathbf{l}} = \{0, \infty\}$	$\tilde{\pi}_0^{\mathbf{l}} = \{0, \infty\}$
$L_0^{\mathbf{l}} = \{0\}$	$L_{\infty}^{\mathbf{l}} = \{\infty\}$
$X_{(\mathbf{l}, 0)} = \mathbb{R} \times \{0\}$	$X_{(\mathbf{l}, \infty)} = (\mathbb{R} \times (\mathbb{R} \setminus \{0\})) \cup \{\infty\}$

6. RELATIONS BETWEEN LIMIT AND END SPACES OF A FLOW AND ITS DYNAMICAL PROPERTIES

6.1. Periodic points. The relation of the limit space of a flow or an \mathbf{r} -exterior flow and the subflow of periodic points is analysed in the following results:

Lemma 6.1. *If X is an \mathbf{r} -exterior flow, then $P(X) \subset L(X)$. In particular, if X is a flow, then $P(X) \subset L^{\mathbf{r}}(X)$.*

Proof. Take x a periodic point and $E \in \varepsilon(X)$ arbitrary. Then there exists $T \in \mathbf{r}$ such that $T \cdot x \subset E$. Since x is periodic, $T \cdot x = \mathbb{R} \cdot x$ and taking into account that $x \in \mathbb{R} \cdot x$, we have that $x \in E$. \square

Lemma 6.2. *Let X be a flow and suppose that X is a T_1 -space. Then, for every $x \in X$ the following statements are equivalent:*

- (i) x is a non-periodic point.
- (ii) $X \setminus \{x\}$ is an \mathbf{r} -exterior subset of X .

Proof. (i) implies (ii): Take $y \in X$; if the trajectory of y is different of the trajectory of x , then for every $T \in \mathbf{r}$, $T \cdot y \subset X \setminus \{x\}$. If y is in the trajectory of x , considering that x is not periodic, one can find $T \in \mathbf{r}$ such that $T \cdot y \subset X \setminus \{x\}$. Then, one has that $X \setminus \{x\} \in \varepsilon^{\mathbf{r}}(X)$. Conversely, suppose that x is a periodic point, by Lemma 6.1 above $X \setminus \{x\}$ is not \mathbf{r} -exterior. \square

Theorem 6.3. *Let X be a flow and suppose that X is a T_1 -space. Then*

$$P(X) = L^r(X).$$

Proof. Let $x \in X \setminus P(X)$. Then, by Lemma 6.2, one has that $X \setminus \{x\} \in \varepsilon^r(X)$ and

$$P(X) = X \setminus \left(\bigcup_{x \notin P(X)} \{x\} \right) = \bigcap_{x \notin P(X)} X \setminus \{x\} \supset \bigcap_{E \in \varepsilon^r(X)} E = L^r(X).$$

Now the results follows from Lemma 6.1. \square

Taking into account the result above, if X is flow and X is a T_1 space we also have that

$$L^r(X) = P(X) \subset P^r(X) \subset \Omega^r(X) \subset X.$$

6.2. Limit spaces and invariant sets.

Lemma 6.4. *Given a flow $\varphi: \mathbb{R} \times X \rightarrow X$ and $A \subset X$ we have that*

$$\text{inv}(A) = \bigcap_{r \in \mathbb{R}} \varphi_r(A).$$

Proof. If $p \in \text{inv}(A)$, then $\mathbb{R} \cdot p \subset A$. Notice that $p = \varphi_r(\varphi_{-r}(p)) \in \varphi_r(A)$ so $p \in \bigcap_{r \in \mathbb{R}} \varphi_r(A)$. Conversely, if $p \in \bigcap_{r \in \mathbb{R}} \varphi_r(A)$, then $p = \varphi_r(x_r)$, $x_r \in A$. This implies that $\varphi_{-r}(p) = x_r \in A$ and $p \in \text{inv}(A)$. \square

Then, by Proposition 4.3, we obtain:

Proposition 6.5. *If X is an \mathbf{r} -exterior flow, then*

- (i) $L(X) = \lim_{E \in \varepsilon(X)} E = \lim_{E \in \varepsilon(X)} \text{inv}(E)$,
- (ii) $\bar{L}(X) = \lim_{E \in \varepsilon(X)} \bar{E} = \lim_{E \in \varepsilon(X)} \text{inv}(\bar{E})$.

Next we give a characterization of the points lying in the difference $\bar{L}(X) \setminus L(X)$:

Proposition 6.6. *Let X be an \mathbf{r} -exterior flow and $x \in \bar{L}(X)$. Then $x \in \bar{L}(X) \setminus L(X)$ if and only if there exist $E \in \varepsilon(X)$ and $t \in \mathbb{R}$ such that $t \cdot x \in \text{Fr}(E)$.*

Note that propositions above can be applied to $L^r(X)$ and $\bar{L}^r(X)$ for a continuous flow X .

6.3. Limits and Ω -limits. In the following result, we analyse the relationship between Ω^r -limit and the bar-limit induced by an externology.

Lemma 6.7. *If X is an \mathbf{r} -exterior flow, then*

$$\Omega^r(X) \subset \bar{L}(X)$$

Proof. If $E \in \varepsilon(X)$, for every $x \in X$ there exist $T \in \mathbf{r}$ such that $T \cdot x \subset E$. Therefore $\overline{T \cdot x} \subset \bar{E}$. By definition this implies that $\omega^r(x) \subset \bar{L}(X)$ for every $x \in X$. Hence $\Omega^r(X) \subset \bar{L}(X)$. \square

Proposition 6.8. *Let X be an \mathbf{r} -exterior flow, $\varepsilon(X)$ its externology and $x \in X$. Then there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \bar{V}_x \in \varepsilon(X)$ if and only if $x \notin \bar{L}(X)$.*

Proof. Suppose that $X \setminus \bar{V}_x \in \varepsilon(X)$. Since $V_x \cap (X \setminus \bar{V}_x) = \emptyset$, then $x \notin \overline{X \setminus \bar{V}_x}$. Since $X \setminus \bar{V}_x \in \varepsilon(X)$, it follows that $x \notin \bigcap_{E \in \varepsilon(X)} \bar{E} = \bar{L}(X)$.

Conversely, if $x \notin \bar{L}(X)$, then there is $E \in \varepsilon(X)$ such that $x \notin \bar{E}$. Taking $V_x = X \setminus \bar{E} = \text{int}(X \setminus E)$, we have that $X \setminus \bar{V}_x = \text{int}(X \setminus V_x) = \text{int}(\bar{E}) \supset E$. Consequently, $X \setminus \bar{V}_x \in \varepsilon(X)$. \square

Corollary 6.9. *Let X be an \mathbf{r} -exterior flow, $\varepsilon(X)$ its externology and $x \in X$. If there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \bar{V}_x \in \varepsilon(X)$, then $x \notin \overline{\Omega^{\mathbf{r}}(X)}$.*

Proof. It is a consequence of Proposition 6.8 and Lemma 6.7. \square

Corollary 6.10. *Let X be a flow and $x \in X$. If there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \bar{V}_x$ is \mathbf{r} -exterior, then $x \notin \overline{\Omega^{\mathbf{r}}(X)}$.*

Lemma 6.11. *Let X be a flow and X is a locally compact regular space. If $x \notin \overline{\Omega^{\mathbf{r}}(X)}$, then there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \bar{V}_x$ is \mathbf{r} -exterior.*

Proof. Suppose that $x \notin \overline{\Omega^{\mathbf{r}}(X)}$. Since X is locally compact, there is a compact neighborhood K at x such that $K \cap \Omega^{\mathbf{r}}(X) = \emptyset$. Take $y \in X$ and assume that for every $T \in \mathbf{r}$, $T \cdot y \cap K \neq \emptyset$. Then there is a sequence $t_n \rightarrow +\infty$ such that $t_n \cdot y \in K$. Since K is compact, one can take a subsequence $t_{n_i} \rightarrow +\infty$ such that $t_{n_i} \cdot y \rightarrow u \in K$. This fact implies that $u \in K \cap \omega^{\mathbf{r}}(y) \subset K \cap \Omega^{\mathbf{r}}(X)$, which is a contradiction. Therefore, there is T such that $T \cdot y \cap K = \emptyset$. Since X is regular, one has that there is $V_x \in (\mathbf{t}_X)_x$ such that $\bar{V}_x \subset K$ and $X \setminus \bar{V}_x$ is \mathbf{r} -exterior. \square

Corollary 6.12. *Let X be a flow. If X is a locally compact regular space, then $\bar{L}^{\mathbf{r}}(X) \subset \overline{\Omega^{\mathbf{r}}(X)}$.*

Proof. If $x \notin \overline{\Omega^{\mathbf{r}}(X)}$, by lemma above, there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \bar{V}_x$ is \mathbf{r} -exterior. By Proposition 6.8, it follows that $x \notin \bar{L}^{\mathbf{r}}(X)$. \square

Theorem 6.13. *Let X be a flow. If X is a locally compact regular space, then $\bar{L}^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$.*

Proof. By corollary above $\bar{L}^{\mathbf{r}}(X) \subset \overline{\Omega^{\mathbf{r}}(X)}$ and by Lemma 6.7, $\Omega^{\mathbf{r}}(X) \subset \bar{L}(X)$. \square

Corollary 6.14. *Let X be a flow. If X is a locally compact T_3 space, then $L^{\mathbf{r}}(X) = P(X)$, $\bar{L}^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$ and*

$$L^{\mathbf{r}}(X) = P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X) \subset \overline{\Omega^{\mathbf{r}}(X)} = \bar{L}^{\mathbf{r}}(X).$$

Proof. It is a consequence of Theorems 6.3 and 6.13. \square

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J.M. GARCÍA CALCINES
 DEPARTAMENTO DE MATEMÁTICA FUNDAMENTAL
 UNIVERSIDAD DE LA LAGUNA
 38271 LA LAGUNA.
E-mail address: jmgarcal@ull.es

L. HERNÁNDEZ PARICIO; M. TERESA RIVAS RODRÍGUEZ
 DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN
 UNIVERSIDAD DE LA RIOJA
 26004 LOGROÑO.
E-mail address: luis-javier.hernandez@unirioja.es
E-mail address: maria-teresa.rivas@unirioja.es